# Special representations of $\mathcal{U}_q(sl(N))$ at the roots of unity



1996 J. Phys. A: Math. Gen. 29 1201

(http://iopscience.iop.org/0305-4470/29/6/008)

J. Phys. A: Math. Gen. **29** (1996) 1201–1214. Printed in the UK

# Special representations of $\mathcal{U}_q(sl(N))$ at the roots of unity

Boucif Abdesselam†
Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France‡

Received 18 September 1995, in final form 24 November 1995

**Abstract.** We show how to adapt the Gelfand–Zetlin basis for describing the special representations of $\mathcal{U}_q(sl(N))$ when $q$ is a root of unity. The explicit construction of special representations is presented in detail for $N = 3$.

## 1. Introduction

The present paper is a companion paper to [1], in which we presented an improvement of [2] giving Gelfand–Zetlin construction of irreducible representations of $\mathcal{U}_q(sl(N))$ at roots of unity independently of their nature. We have shown that it is possible to describe the periodic, semi-periodic, nilpotent, usual and some special representations of $\mathcal{U}_q(sl(N))$ by the fractional parts formalism. However, special representations generally need a special treatment.

In this paper, we restrict ourselves to the quantum Lie algebra, where the raising and lowering operators are nilpotent, i.e. $e_i^m = f_i^m = 0$ and where the Cartan generators $h_i$ are such that $k_i^m = \left(q^{h_i}\right)^m = 1$ (representations of this case were studied by Luszting [3]). A classification of irreducible representations of $\mathcal{U}_q(sl(3))$ was done by Dobrev in [4, 5].

The Gelfand–Zetlin basis in the form [1] is not yet totally adapted for special representations. Note that the paper [7] provides the special representations of $\mathcal{U}_q(sl(3))$ because the matrix elements do not contain denominators, and of course do not generate divergence when $q$ is a root of unity.

Our purpose is to provide a procedure that enables the construction of general special representations by a suitable adaptation of the Gelfand–Zetlin basis.

In section 2, we give the general idea for the construction of special representations of $\mathcal{U}_q(sl(N))$. Explicit construction of the special representations of $\mathcal{U}_q(sl(3))$ and an example of explicit construction of flat representations based in the Gelfand–Zetlin pattern are presented in section 3.

## 2. The Primitive Gelfand–Zetlin basis

### 2.1. The quantum algebra $\mathcal{U}_q(sl(N))$

The quantum algebra $\mathcal{U}_q(sl(N))$ [8, 9] is defined by the generators $k_i$, $k_i^{-1}$, $e_i$, $f_i$ ($i = 1, \ldots, N-1$) and the relations

† E-mail address: abdess@orphee.polytechnique.fr
‡ Laboratoire Propre du CNRS UPR A.0014.

0305-4470/96/061201+14$19.50 

$$\begin{aligned} &k_i e_j = q^{a_{ij}} e_j k_i \qquad k_i f_j = q^{-a_{ij}} f_j k_i \\ &[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \\ &[e_i, e_j] = 0 \qquad \text{for} \quad |i-j| > 1 \\ &e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0 \\ &f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0 \,. \end{aligned} \tag{2.1}$$

The two last equations are called the Serré relations, and $(a_{ij})_{i,j=1,\ldots,N-1}$ is the Cartan matrix of $sl(N)$, i.e.

$$a_{ij} = \begin{cases} 2 & \text{for} \quad i = j \\ -1 & \text{for} \quad j = i \pm 1 \\ 0 & \text{for} \quad |i-j| > 1\,. \end{cases} \tag{2.2}$$

Let us now define the adapted Gelfand–Zetlin basis for the representations of $\mathcal{U}_q(sl(N))$, the corresponding states are called *primitive vectors* of Gelfand–Zetlin pattern.

### 2.2. *Primitive vectors of the Gelfand–Zetlin basis*

The states are

$$|p\rangle = \left| \begin{matrix} p_{1N} & & p_{2N} & \cdots & p_{N-1,N} & & p_{NN} \\ & p_{1N-1} & & \cdots & & p_{N-1,N-1} & \\ & & \ddots & \cdots & \iddots & & \\ & & p_{12} & & p_{22} & & \\ & & & p_{11} & & & \end{matrix} \right\rangle \tag{2.3}$$

(with respect to [2], we use $p_{ij} = h_{ij} - i$ instead of $h_{ij}$). The primitive Gelfand–Zetlin basis [10] is labelled by $\frac{1}{2}N(N+1)$ numbers $p_{ij}$. The first line of indices determines the *highest weight* of the representations, whereas the others move by steps of $\pm 1$ under the action of the raising and lowering generators. The whole set of $p_{il}$'s is defined up to an overall constant. One can constrain, for example, $\sum p_{iN}$, or $p_{NN}$, to be zero. For the classical case and when $q$ is generic, the states (2.3) within the same module $V([p]_N)$ are distinguished by $p_{ij}$, $i, j = 1, \ldots, N-1$, which assume values consistent with the triangular inequalities

$$\begin{aligned} &p_{i,j+1} - p_{i,j} \in Z_+ \\ &p_{i,j} - p_{i+1,j+1} - 1 \in Z_+ \qquad i, j = 1, \ldots, N-1 \end{aligned} \tag{2.4}$$

or

$$p_{i,j+1} \geqslant p_{ij} > p_{i+1,j+1} \,. \tag{2.5}$$

The dimension of $V([p]_N)$ is given by

$$\dim V([p]_N) = \frac{\prod_{i=1}^{N-1} \prod_{j=i+1}^{N} (p_{iN} - p_{jN})}{\prod_{i=1}^{N-1} (N-i)!} \,. \tag{2.6}$$

### 2.3. *The primitive representations*

The action of the generators $k_l^{\pm 1}$, $e_l$ and $f_l$ $(l = 1, 2, \ldots, N-1)$ is given by

$$
\begin{aligned}
k_l^{\pm 1}|p\rangle &= q^{\pm(2\sum_{i=1}^{l} p_{il} - \sum_{i=1}^{l+1} p_{i,l+1} - \sum_{i=1}^{l-1} p_{i,l-1} - 1)}|p\rangle \\
f_l|p\rangle &= \sum_{j=1}^{l} \frac{P_1(jl;\,p)P_2(jl;\,p)}{P_3(jl;\,p)}|p_{jl}-1\rangle \\
e_l|p\rangle &= \sum_{j=1}^{l} \frac{P_1(jl;\,p_{jl}+1)P_2(jl;\,p_{jl}+1)}{P_3(jl;\,p_{jl}+1)}|p_{jl}+1\rangle
\end{aligned}
\tag{2.7}
$$

where $|p_{jl} \pm 1\rangle$ denotes the state differing from $|p\rangle$ by only $p_{jl} \to p_{jl} \pm 1$, and

$$
\begin{aligned}
P_1(jl;\,p) &= \prod_{i=1}^{l+1} [\varepsilon_{ij}(p_{i,l+1} - p_{j,l} + 1)]^{1/2} \\
P_2(jl;\,p) &= \prod_{i=1}^{l-1} [\varepsilon_{ji}(p_{j,l} - p_{i,l-1})]^{1/2} \\
P_3(jl;\,p) &= \prod_{\substack{i=1 \\ i\neq j}}^{l} [\varepsilon_{ij}(p_{i,l} - p_{j,l})]^{1/2}[\varepsilon_{ij}(p_{i,l} - p_{j,l} + 1)]^{1/2}
\end{aligned}
\tag{2.8}
$$

$\varepsilon_{ij}$ being the sign defined by

$$
\varepsilon_{ij} = \begin{cases} 1 & \text{for} \quad i \leqslant j \\ -1 & \text{for} \quad i > j\,. \end{cases}
\tag{2.9}
$$

In the following, we take $q$ to be a root of unity and $p_{ij}$ as *integers*. Let $m$ be the smallest integer such that $q^m = 1$. We will consider only the case of odd $m$ in this paper. A similar discussion is valid when $m$ is even.

Here we consider *the quantum analogue of classical (highest weight and lowest weight) irreducible representations* with a highest weight that obeys

$$
p_{1N} - p_{NN} > m\,.
\tag{2.10}
$$

When $q^m = 1$ these representations are not always irreducible, since some new *singular vectors* arise in the corresponding Verma module, that are not obtained from the highest weight vector by the action of the finite-dimensional Weyl group. Quotienting by the sub-representations generated by the singular vectors leads to new irreducible representations that we call *special representations*.

### 2.4. *Adaptation of the Gelfand–Zetlin basis and special representations*

Let $\eta_{jl}$ and $\eta'_{jl}$ be, respectively, the numbers of zeros of $P_1(jl;\,p)$, $P_2(jl;\,p)$ and $P_3(jl;\,p)$. We note that the maximum value of $\eta'_{jl}$ is $l-1$. If a primitive vector from the right-hand side of (2.7) does not belong to the module under consideration, then the corresponding term is zero ($\eta_{jl} > \eta'_{jl}$). If an equal number of factors in the numerators and denominators are simultaneously equal to zero ($\eta_{jl} = \eta'_{jl}$), they should be cancelled out and the corresponding primitive vector here belongs to the module. If $\eta'_{jl} > \eta_{jl}$, the matrix elements of $f_l$ are undefined. Next, we will show how to eliminate the divergences of the matrix elements using a change of bases. This method will be illustrated on a simple example in section 3.

Suppose

$$\eta_{jl} = 0 \qquad \text{and} \qquad \eta'_{jl} = l - 1 \qquad 1 \leqslant j \leqslant l \tag{2.11}$$

i.e.

$$\begin{aligned} &p_{i,l+1} - p_{jl} + 1 \neq 0\ [m] \qquad 1 \leqslant i \leqslant l+1 \qquad 1 \leqslant j \leqslant l \\ &p_{jl} - p_{i,l-1} \neq 0\ [m] \qquad 1 \leqslant j \leqslant l \qquad 1 \leqslant i \leqslant l-1 \\ &p_{il} - p_{jl} = 0\ [m] \qquad 1 \leqslant j \leqslant l \qquad 1 \leqslant i \leqslant l \end{aligned} \tag{2.12}$$

thus, there exist $\beta_1, \beta_2, \ldots, \beta_{l-1} \in Z_+$ such that

$$\begin{aligned} &p_{i,l} - p_{i+1,l} = \beta_i m \qquad 1 \leqslant i \leqslant l-1 \\ &p_{i,l} - p_{j,l} = (\beta_i + \beta_{i+1} + \cdots + \beta_{j-1})m \qquad j > i\,. \end{aligned} \tag{2.13}$$

The action of $f_l$ over a state satisfying (2.12) produces in the right-hand side of (2.7) a set of states $\{|p'_{1l} \ldots p'_{il} \cdots p_{ll}\rangle, 1 \leqslant i \leqslant l\}$, where

$$\begin{aligned} p'_{1l} &= \begin{cases} p_{il} + (\beta_1 + \cdots + \beta_{i-1})m & \text{for} \quad i > 1 \\ p_{1l} - 1 & \text{for} \quad i = 1 \end{cases} \\ p'_{il} &= \begin{cases} p_{1l} - 1 - (\beta_1 + \cdots + \beta_{i-1})m & \text{for} \quad i > 1 \\ p_{1l} - 1 & \text{for} \quad i = 1 \end{cases} \end{aligned} \tag{2.14}$$

this set is called a set of *type* $p_{1l} - 1$ or $(\{1\}, \{2, \ldots, l\})$.

*Definition 1.* Let a state satisfy

$$\begin{aligned} &p_{1l} - p_{il} \pm \beta = 0\ [m], i = 2, \ldots, l \qquad 1 \leqslant |\beta| \leqslant m-1 \\ &p_{il} - p_{jl} = 0\ [m] \qquad i, j = 2, \ldots, l\,. \end{aligned} \tag{2.15}$$

This state is called a state of *type* $p_{1l}$ *or* $(\{1\}, \{2, \ldots, l\})$. Define the operation

$$\pi^l_{1i}(|p_{1l} \ldots p_{il} \ldots p_{ll}\rangle) = |p'_{1l} \ldots p'_{il} \ldots p_{ll}\rangle \tag{2.16}$$

where

$$p'_{1l} = \begin{cases} p_{il} + (\beta_1 + \cdots + \beta_{i-1})m & \text{for} \quad i > 1 \\ p_{1l} & \text{for} \quad i = 1 \end{cases} \tag{2.17}$$

$$p'_{il} = \begin{cases} p_{1l} - (\beta_1 + \cdots + \beta_{i-1})m & \text{for} \quad i > 1 \\ p_{1l} & \text{for} \quad i = 1\,. \end{cases} \tag{2.18}$$

This operation is called *exchange mapping of level* $l$ *centred on 1*, and the set $\{|p'_{1l} \ldots p'_{il} \ldots p_{ll}\rangle, 1 \leqslant i \leqslant l\}$ is of *type* $p_{1l}$ *or* $(\{1\}, \{2, \ldots, l\})$. This set of states has the same eigenvalues for the Cartan operators (degenerate states) and has to satisfy the triangular inequalities.

*Lemma*. Let a state satisfy the condition (2.15), and let $\{|p'_{1l}\ldots p'_{il}\cdots p_{ll}\rangle, 1\leqslant i\leqslant l\}$ be the set of all states obtained by action of the mapping $\pi^l_{1i}$ over this state. This set is *isomorphic* to a set obtained by action of the mapping $\pi^l_{\mu\nu}$ over a state of *type* $p_{\mu l}\mp\beta(\mu\neq 1)$.

*Proof*. Let a state be of type $(\{1\},\{2,\ldots,l\})$, i.e.

$$p_{1l}-p_{il}\pm\beta=0\ [m]\qquad i=2,\ldots,l\qquad 1\leqslant|\beta|\leqslant m-1$$
$$p_{il}-p_{jl}=0\ [m]\qquad i\neq 1\quad\text{and}\quad j\neq 1$$

and let

$$\pi^l_{1\mu}(|p_{1l}\ldots p_{\mu l}\ldots p_{ll}\rangle)=|p_{1l}\pm\beta\ldots p_{\mu l}\mp\beta\ldots p_{ll}\rangle\,.$$

The state in the right-hand side is of *type* $p_{\mu l}\mp\beta(\mu\neq 1)$, and

$$\pi^l_{\mu\nu}\circ\pi^l_{1\mu}=\pi^l_{1\nu}\,. \tag{2.19}$$

All states of the set (2.14) are obtained by action of the mapping $\pi^l_{1i}$ over the state $|p_{1l}-1\ldots p_{il}\ldots p_{ll}\rangle$. Using this notation, the action of $f_l$ is

$$f_l|p\rangle=\sum_{j=1}^{l}\frac{P_1(jl;p)P_2(jl;p)}{P_3(jl;p)}|p'_{1l}\ldots p'_{jl}\ldots p_{ll}\rangle\,. \tag{2.20}$$

We note that the number of zeros in the polynomials $P_3(jl;p)$ is $l-1$.

*Definition 2*. Let a set be a state of *type* $(\{1\},\{2,\ldots,l\})$, i.e.

$$p_{1l}-p_{il}\pm\beta=0\ [m]\qquad 2\leqslant i\leqslant l\qquad 1\leqslant|\beta\leqslant m-1$$

and let the new basis be given by

$$|p'_{1l}\ldots p'_{il}\ldots p_{ll}\rangle=\sum_{j=1}^{l}D_{ij}\|p'_{1l}\ldots p'_{jl}\ldots p_{ll}\rangle \tag{2.21}$$

where $D$ is a $l\times l$ rotation matrix, i.e.

$$D^t\cdot D=D\cdot D^t=\mathbf{1}\,. \tag{2.22}$$

This new basis is called the *modified basis of type* $(\{1\},\{2,\ldots,l\})$. The primitive and the modified sets satisfy the triangular inequalities (2.5).

The finiteness of the matrix elements $\langle p|e_lf_l|p\rangle$ and $\langle p|f_le_l|p\rangle$ (preserve $[e_l,f_l]=\frac{k_l-k_l^{-1}}{q-q^{-1}}$) imply that there exists a modified basis such that the new matrix elements are without divergences. Using this definition, the equation (2.20) is reduced to

$$\begin{aligned}f_l|p\rangle&=\sum_{i=1}^{l}\left(\sum_{j=1}^{l}\frac{P_1(jl;p)P_2(jl;p)}{P_3(jl;p)}D_{ji}\right)\|p'_{1l}\ldots p'_{il}\ldots p_{ll}\rangle\\&=\sum_{i=1}^{l}A_{il}\|p'_{1l}\ldots p'_{il}\ldots p_{ll}\rangle\end{aligned} \tag{2.23}$$

where, $A_{il}$ are the new matrix elements associated to the modified basis, i.e.

$$A_{il} = \sum_{j=1}^{l} \frac{P_1(jl;\,p)P_2(jl;\,p)}{P_3(jl;\,p)} D_{ji} \qquad 1 \leqslant i \leqslant l\,. \tag{2.24}$$

These matrix elements are called the *modified matrix elements*. Generally, we choose

$$\begin{aligned} A_{1l} &= \left(\sum_{j=1}^{l} \frac{P_1^2(jl;\,p)P_2^2(jl;\,p)}{P_3^2(jl;\,p)}\right)^{1/2} \\ A_{il} &= 0 \qquad 2 \leqslant i \leqslant l \end{aligned} \tag{2.25}$$

i.e.

$$f_l|p\rangle = \left(\sum_{j=1}^{l} \frac{P_1^2(jl;\,p)P_2^2(jl;\,p)}{P_3^2(jl;\,p)}\right)^{1/2} \| p_{1l} - 1 \dots p_{ll}\rangle\,. \tag{2.26}$$

We note that the matrix element of (2.26) is finite (i.e. without divergences).

We are now able to claim this generalization,

*Definition 3.* Let a collection $\{\boldsymbol{I}_k, k \in \boldsymbol{J} \subset \boldsymbol{N}\}$ of subsets of $\{1, \dots, l\}$ satisfying the following conditions:

$$\begin{aligned} &\text{(i)} && \text{if} \quad k < s \qquad \forall i \in \boldsymbol{I}_k \quad \forall j \in \boldsymbol{I}_s \quad \text{thus} \quad i < j \\ &\text{(ii)} && \boldsymbol{I}_k \cap \boldsymbol{I}_s = \emptyset \qquad \text{if} \quad k \neq s \\ &\text{(iii)} && \bigcup_{k \in \boldsymbol{J}} \boldsymbol{I}_k = \{1, \dots, l\} \end{aligned} \tag{2.27}$$

and where

$$\begin{aligned} p_{il} - p_{jl} &= [m] \qquad i, j \in \boldsymbol{I}_k \quad i < j \\ p_{il} - p_{jl} &= \zeta_{ks}[m] \qquad i \in \boldsymbol{I}_k \quad j \in \boldsymbol{I}_s \quad k < s \quad 1 \leqslant |\zeta_{ks}| \leqslant m - 1\,. \end{aligned} \tag{2.28}$$

This state is called a state of *type* $(\boldsymbol{I}_k, k \in \boldsymbol{J} \subset \boldsymbol{N})$. Let $\pi^l_{ks;ij}$ be the *exchange mapping of level $l$ between the subsets $\boldsymbol{I}_k$ and $\boldsymbol{I}_s$* $(k < s)$, i.e. if

$$p_{il} - p_{jl} = \zeta_{ks} + (\beta_i + \beta_{i+1} + \dots + \beta_{j-1})m \tag{2.29}$$

we have

$$\pi^l_{ks;ij}(|p_{1l} \dots p_{il} \dots p_{jl} \dots p_{ll}\rangle) = |p_{1l} \dots p'_{il} \dots p'_{jl} \dots p_{ll}\rangle \tag{2.30}$$

with

$$\begin{aligned} p'_{il} &= p_{jl} + (\beta_i + \beta_{i+1} + \dots + \beta_{j-1})m = p_{il} + \zeta_{ks} \\ p'_{jl} &= p_{il} - (\beta_i + \beta_{i+1} + \dots + \beta_{j-1})m = p_{jl} - \zeta_{ks}\,. \end{aligned} \tag{2.31}$$

The operators $k_i$ commutes with exchange mapping $\pi^l_{ks;ij}$. Let $|p\rangle$ be the state satisfying (2.28) and (2.29). Define

$$V_l(\boldsymbol{I}_k, k \in \boldsymbol{J}) = \left\{ \prod_{\substack{k_\alpha \neq s_\alpha \\ i_\alpha \neq j_\alpha}} \pi_{k_\alpha s_\alpha; i_\alpha j_\alpha}(|p_{1l} \ldots p_{il} \ldots p_{jl} \ldots p_{ll}\rangle) \right\}. \tag{2.32}$$

The set $V_l(\boldsymbol{I}_k, k \in \boldsymbol{J} \subset \boldsymbol{N})$ is obtained by all possible changes between the different subsets $\boldsymbol{I}_k (k \in \boldsymbol{J})$. We note that the all states of $V_l(\boldsymbol{I}_k, k \in \boldsymbol{J} \subset \boldsymbol{N})$ have the same eigenvalues for the Cartan operators.

For example,

$$\boldsymbol{J} = \{1\} \qquad \boldsymbol{I}_1 = \{1, \ldots, l\} \qquad \dim V_l = 1$$

$$\boldsymbol{J} = \{1, 2\} \qquad \boldsymbol{I}_1 = \{1\} \qquad \boldsymbol{I}_2 = \{2, \ldots, l\} \qquad \dim V_l = l$$

$$\boldsymbol{J} = \{1, 2\} \qquad \boldsymbol{I}_1 = \{1, 2\} \qquad \boldsymbol{I}_2 = \{3, \ldots, l\} \qquad \dim V_l = \tfrac{1}{2} l(l-1)\,.$$

*Definition 4.* Let a new basis given by

$$|p'_{1l} \ldots p'_{il} \ldots p_{ll}\rangle = \sum_{j=1}^{l} D_{ij} \| p'_{1l} \ldots p'_{jl} \ldots p_{ll}\rangle \tag{2.33}$$

where the states $|p'_{1l} \ldots p'_{il} \ldots p_{ll}\rangle$ are in the set $V_l(\boldsymbol{I}_k, k \in \boldsymbol{J} \subset \boldsymbol{N})$ and $D$ is a $\dim V_l \times \dim V_l$ rotation matrix. This new basis is called the *modified basis of type* $(\boldsymbol{I}_k, k \in \boldsymbol{J} \subset \boldsymbol{N})$ . We note that the primitive and the modified bases satisfy the triangular inequalities. The set of states $\| p'_{1l} \ldots p'_{jl} \ldots p_{ll}\rangle$ is denoted by $\overline{V}_l(\boldsymbol{I}_k, k \in \boldsymbol{J} \subset \boldsymbol{N})$.

Using definition 3, the action of $f_l$ over a modified state of *type* $(\{1\}, \{2, \ldots, l\})$ produces two sets of states, respectively, of *type* $(\{1\}, \{2, \ldots, l\})$ and $(\{1, 2\}, \{3, \ldots, l\})$. The matrix elements of $(\{1\}, \{2, \ldots, l\}) \to (\{1\}, \{2, \ldots, l\})$ do not contain divergences (see the structure of $P_3(jl; p)$). But if the matrix elements of $(\{1\}, \{2, \ldots, l\}) \to (\{1, 2\}, \{3, \ldots, l\})$ contain some divergences, using definition 4 we take a rotation in this second set in such a way as to eliminate these divergences ($\langle p|e_l f_l|p\rangle$ and $\langle p|f_l e_l|p\rangle$ have to remain finite). We have to repeat this mechanism as far as the elimination of all divergences of the Gelfand–Zetlin representations.

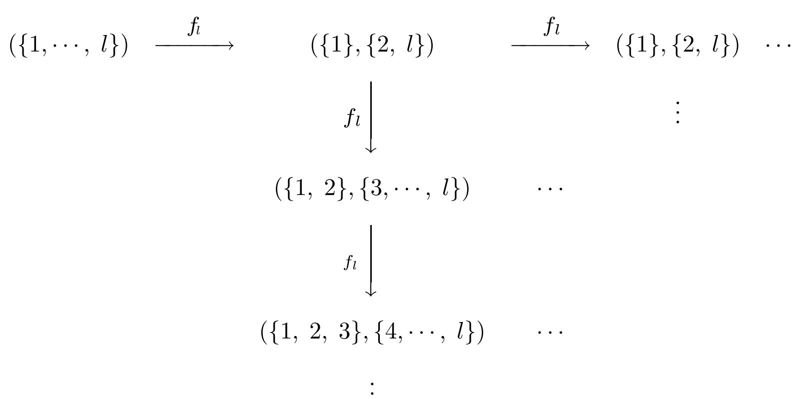


**Figure 1.** Different transitions between the types of sets in the case of $\mathcal{U}_q(sl(N))$.

## 3. Applications

### *3.1. Explicit construction of the special representations*

In the following, we present in detail the explicit construction of special representations for $N = 3$. We will consider only the operators $f_1$ and $f_2$, a similar discussion is valid for $e_1$ and $e_2$. The primitive Gelfand–Zetlin state for this particular case is just

$$|p\rangle = \left| \begin{matrix} p_{13} & & p_{23} & & p_{33} \\ & p_{12} & & p_{22} & \\ & & p_{11} & & \end{matrix} \right\rangle \tag{3.1}$$

where $p_{33}$ is chosen to be equal to zero. The actions of the generators $f_1$ and $f_2$ for $\mathcal{U}_q(sl(3))$ are given by

$$f_1|p\rangle = ([p_{12} - p_{11} + 1][p_{11} - p_{22} - 1])^{1/2}|p_{11} - 1\rangle \tag{3.2}$$

$$\begin{aligned} f_2|p\rangle = {} & \left(\frac{[p_{13} - p_{12} + 1][p_{12} - p_{23} - 1][p_{12} - p_{33} - 1][p_{12} - p_{11}]}{[p_{12} - p_{22}][p_{12} - p_{22} - 1]}\right)^{1/2} |p_{12} - 1\rangle \\ & + \left(\frac{[p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{22} - p_{33} - 1][p_{11} - p_{22}]}{[p_{12} - p_{22}][p_{12} - p_{22} + 1]}\right)^{1/2} |p_{22} - 1\rangle\,. \end{aligned} \tag{3.3}$$

We note that in this case there are only two types of sets $(\{1, 2\})$ and $(\{1\}, \{2\})$ and the maximal number of zeros in the denominator of the matrix elements is only one.

Now suppose that

$$\begin{aligned} & p_{i3} - p_{j2} + 1 \not\equiv 0\ [m] \qquad 1 \leqslant i \leqslant 3 \qquad \text{and} \qquad 1 \leqslant j \leqslant 2 \\ & p_{i2} - p_{11} \not\equiv 0\ [m] \qquad 1 \leqslant i \leqslant 2 \end{aligned} \tag{3.4}$$

i.e.

$$\eta_{12} = \eta_{22} = 0\,. \tag{3.5}$$

The matrix elements of $f_2$ are infinite if

$$\begin{aligned} & \text{(i)} \qquad p_{12} - p_{22} = 0\ [m] \\ & \text{(ii)} \qquad p_{12} - p_{22} + 1 = 0\ [m] \\ & \text{(iii)} \qquad p_{12} - p_{22} - 1 = 0\ [m]\,. \end{aligned} \tag{3.6}$$

*Case (a).* Let a state satisfy the following conditions:

$$p_{12} - p_{22} = 0\ [m] \qquad \text{or} \qquad p_{12} - p_{22} = \beta m \qquad \beta \in Z_+ \tag{3.7}$$

i.e.

$$\eta'_{12} = \eta'_{22} = 1\,. \tag{3.8}$$

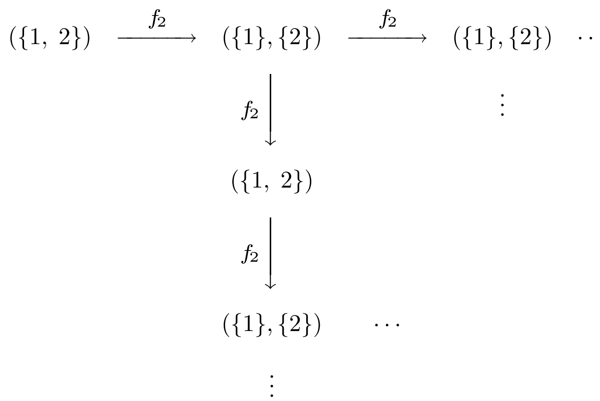


**Figure 2.** Different transitions between the types of sets $(\{1, 2\})$ and $(\{1\}, \{2\})$ in the case of $\mathcal{U}_q(sl(3))$.

The action of $f_2$ on this state give

$$f_2|p_{12}\ \ p_{22}\rangle = \frac{\kappa}{([m][m-1])^{1/2}}|p_{12}-1\ \ p_{22}\rangle + \frac{\kappa}{([m][m+1])^{1/2}}|p_{12}\ \ p_{22}-1\rangle \tag{3.9}$$

where

$$\kappa = ([p_{13}-p_{22}+1][p_{23}-p_{22}+1][p_{22}-p_{33}-1][p_{11}-p_{22}])^{1/2}\,. \tag{3.10}$$

Using definitions 3 and 4, the relation between the primitive and the modified states is

$$\begin{pmatrix} |p_{12}-1\ \ p_{22}\rangle \\ |p_{22}+\beta m\ \ p_{12}-1-\beta m\rangle \end{pmatrix} = D(\phi)\begin{pmatrix} \|p_{12}-1\ \ p_{22}\rangle \\ \|p_{22}+\beta m\ \ p_{12}-1-\beta m\rangle \end{pmatrix}$$

i.e.

$$\begin{pmatrix} |p_{12}-1\ \ p_{22}\rangle \\ |p_{12}\ \ p_{22}-1\rangle \end{pmatrix} = D(\phi)\begin{pmatrix} \|p_{12}-1\ \ p_{22}\rangle \\ \|p_{12}\ \ p_{22}-1\rangle \end{pmatrix} \tag{3.11}$$

for any values of $p_{11}$ satisfying the triangular inequality, respectively, for the primitive and the modified basis, and where

$$D(\phi) = \begin{pmatrix} \left(\frac{[m-1]}{[2][m]}\right)^{1/2} & \left(\frac{[m+1]}{[2][m]}\right)^{1/2} \\ -\left(\frac{[m+1]}{[2][m]}\right)^{1/2} & \left(\frac{[m-1]}{[2][m]}\right)^{1/2} \end{pmatrix}. \tag{3.12}$$

Using the trivial identity,

$$\frac{[a+1]}{[2][a]} + \frac{[a-1]}{[2][a]} = 1 \tag{3.13}$$

the right-hand side of (3.9) is reduced to

$$f_2|p_{12}\ \ p_{22}\rangle = \left(\frac{[2][p_{13}-p_{22}+1][p_{23}-p_{22}+1][p_{22}-p_{33}-1][p_{11}-p_{22}]}{[p_{12}-p_{22}-1][p_{12}-p_{22}+1]}\right)^{1/2} \times \|p_{11}\ \ p_{22}-1\rangle\,. \tag{3.14}$$

This equation corresponds to the transition $(\{1,2\}) \to (\{1\},\{2\})$.

In this case the action of $f_1$ over the modified Gelfand–Zetlin basis is given by

$$f_1\|p_{12}-1\rangle = \begin{cases} ([p_{12}-p_{11}][p_{11}-p_{22}-1])^{1/2}\|p_{12}-1\rangle_{p_{11}-1} & \text{if} \quad p_{11}\neq p_{22}+1 \\ [m-1]\left(\dfrac{[m+1]}{[2]}\right)^{1/2}|p_{22}-1\rangle_{p_{11}-1} & \text{if} \quad p_{11}=p_{22}+1 \end{cases} \tag{3.15}$$

and

$$f_1\|p_{22}-1\rangle = \begin{cases} ([p_{12}-p_{11}+1][p_{11}-p_{22}])^{1/2}\|p_{22}-1\rangle_{p_{11}-1} & \text{if} \quad p_{11}\neq p_{22}+1 \\ \left(\dfrac{[m-1]}{[2]}\right)^{1/2}|p_{22}-1\rangle_{p_{11}-1} & \text{if} \quad p_{11}=p_{22}+1\,. \end{cases} \tag{3.16}$$

*Remark.* If

$$p_{12}-p_{22}+1=0\ [m] \qquad \text{i.e.} \quad p_{12}-p_{22}+1=\beta m \qquad (\beta\in Z_+) \tag{3.17}$$

the relation between the primitive and the modified basis is given by

$$\begin{pmatrix} |p_{12}\ \ p_{22}\rangle \\ |p_{12}+1\ \ p_{22}-1\rangle \end{pmatrix} = D(\phi)\begin{pmatrix} \|p_{12}\ \ p_{22}\rangle \\ \|p_{12}+1\ \ p_{22}-1\rangle \end{pmatrix} \tag{3.18}$$

with

$$D(\phi) = \begin{pmatrix} \left(\dfrac{[p_{12}-p_{22}]}{[2][p_{12}-p_{22}+1]}\right)^{1/2} & \left(\dfrac{[p_{12}-p_{22}+2]}{[2][p_{12}-p_{22}+1]}\right)^{1/2} \\ -\left(\dfrac{[p_{12}-p_{22}+2]}{[2][p_{12}-p_{22}+1]}\right)^{1/2} & \left(\dfrac{[p_{12}-p_{22}]}{[2][p_{12}-p_{22}+1]}\right)^{1/2} \end{pmatrix}. \tag{3.19}$$

*Case (b).* For a state satisfying the condition

$$p_{12}-p_{22}+1=0\ [m] \qquad \text{i.e.} \quad p_{12}-p_{22}+1=\beta m(\beta\in Z_+) \tag{3.20}$$

the correspondence between the primitive vectors and the modified vectors is given by the following formulae:

$$\begin{pmatrix} \|p_{12}\ \ p_{22}\rangle \\ \|p_{12}+1\ \ p_{22}-1\rangle \end{pmatrix} = D(\phi)\begin{pmatrix} |p_{12}\ \ p_{22}\rangle \\ |p_{12}+1\ \ p_{22}-1\rangle \end{pmatrix} \tag{3.21}$$

where $D(\phi)$ is given by (3.19). If we take a extension of the definition of the modified formula

$$\begin{pmatrix} \|p_{12}-1\ \ p_{22}\rangle \\ \|p_{22}+\beta m\ \ p_{12}-1-\beta m\rangle \end{pmatrix} = D(-\phi)\begin{pmatrix} |p_{12}-1\ \ p_{22}\rangle \\ |p_{22}+\beta m\ \ p_{12}-1-\beta m\rangle \end{pmatrix}$$

i.e.

$$\begin{pmatrix} \|p_{12}-1\ \ p_{22}\rangle \\ \|p_{12}+1\ \ p_{22}-2\rangle \end{pmatrix} = D(-\phi)\begin{pmatrix} |p_{12}-1\ \ p_{22}\rangle \\ |p_{12}+1\ \ p_{22}-2\rangle \end{pmatrix}. \tag{3.22}$$

The action of $f_2$ over the modified states is reduced to

$$f_2\|p_{12}\ \ p_{22}\rangle = \left(\frac{[p_{13}-p_{12}+1][p_{12}-p_{23}-1][p_{12}-p_{33}-1][p_{12}-p_{11}]}{[p_{12}-p_{22}][p_{12}-p_{22}-1]}\right)^{1/2} \times \|p_{12}-1\ \ p_{22}\rangle \tag{3.23}$$

and

$$\begin{aligned} f_2\|p_{12}+1\ \ p_{22}-1\rangle &= \left(\frac{[p_{13}-p_{22}+2][p_{23}-p_{22}+2][p_{22}-p_{33}-2][p_{11}-p_{22}+1]}{[p_{12}-p_{22}+2][p_{12}-p_{22}+3]}\right)^{1/2} \\ &\quad \times \|p_{12}+1\ \ p_{22}-2\rangle \\ &\quad + \left(\frac{[2][p_{13}-p_{22}+1][p_{23}-p_{22}+1][p_{22}-p_{33}-1][p_{11}-p_{22}]}{[p_{12}-p_{22}][p_{12}-p_{22}+2]}\right)^{1/2} \\ &\quad \times |p_{12}\ \ p_{22}-1\rangle\,. \end{aligned} \tag{3.24}$$

We note that the state $|p_{12}\ \ p_{22}-1\rangle$ is of type $(\{1,2\})$. The equations (3.23) and (3.24) correspond, respectively, to the transitions $(\{1\},\{2\}) \to (\{1\},\{2\})$ and $(\{1\},\{2\}) \to (\{1\},\{2\}) + (\{1,2\})$.

*Remark*. For example, the action of $f_2$ over the extension (3.22) gives

$$\begin{aligned} f_2\|p_{12}-1\ \ p_{22}\rangle &= \left(\frac{[p_{13}-p_{12}+2][p_{12}-p_{23}-2][p_{12}-p_{33}-2][p_{12}-p_{11}-1]}{[p_{12}-p_{22}-1][p_{12}-p_{22}-2]}\right)^{1/2} \\ &\quad \times \|p_{12}-2\ \ p_{22}\rangle \\ &\quad + \left(\frac{[p_{13}-p_{22}+1][p_{23}-p_{22}+1][p_{22}-p_{33}-1][p_{11}-p_{22}]}{[p_{12}-p_{22}-1][p_{12}-p_{22}]}\right)^{1/2} \\ &\quad \times \|p_{12}-1\ \ p_{22}-1\rangle \end{aligned} \tag{3.25}$$

where

$$\begin{pmatrix} \|p_{12}-2\ \ p_{22}\rangle \\ \|p_{22}+\beta m\ \ p_{12}-2-\beta m\rangle \end{pmatrix} = D(-\phi) \begin{pmatrix} |p_{12}-2\ \ p_{22}\rangle \\ |p_{22}+\beta m\ \ p_{12}-2-\beta m\rangle \end{pmatrix} \tag{3.26}$$

$$\begin{pmatrix} \|p_{12}-1\ \ p_{22}-1\rangle \\ \|p_{22}-1+\beta m\ \ p_{12}-1-\beta m\rangle \end{pmatrix} = D(-\phi) \begin{pmatrix} |p_{12}-1\ \ p_{22}-1\rangle \\ |p_{22}-1+\beta m\ \ p_{12}-1-\beta m\rangle \end{pmatrix}\,. \tag{3.27}$$

*Case (c)*. The discussion is similar to the case (b). Let a primitive state satisfy the following condition:

$$p_{12}-p_{22}-1 = 0\ [m] \qquad \text{in} \qquad p_{12}-p_{22}-1 = \beta m \qquad (\beta \in Z_+)\,. \tag{3.28}$$

Define

$$\begin{pmatrix} \|p_{12}\ \ p_{22}\rangle \\ \|p_{22}+\beta m\ \ p_{12}-\beta m\rangle \end{pmatrix} = D(-\phi) \begin{pmatrix} |p_{12}-1\ \ p_{22}\rangle \\ |p_{22}+\beta m\ \ p_{12}-\beta m\rangle \end{pmatrix}$$

i.e.

$$\begin{pmatrix} \|p_{12}\ \ p_{22}\rangle \\ \|p_{12}-1\ \ p_{22}+1\rangle \end{pmatrix} = D(-\phi)\begin{pmatrix} |p_{12}\ \ p_{22}\rangle \\ |p_{12}-1\ \ p_{22}+1\rangle \end{pmatrix} \tag{3.29}$$

and take the extension

$$\begin{pmatrix} \|p_{12}\ \ p_{22}-1\rangle \\ \|p_{22}-1+\beta m\ \ p_{12}-\beta m\rangle \end{pmatrix} = D(-\phi)\begin{pmatrix} |p_{12}-1\ \ p_{22}-1\rangle \\ |p_{22}-1+\beta m\ \ p_{12}-\beta m\rangle \end{pmatrix}$$

i.e.

$$\begin{pmatrix} \|p_{12}\ \ p_{22}-1\rangle \\ \|p_{12}-2\ \ p_{22}+1\rangle \end{pmatrix} = D(-\phi)\begin{pmatrix} |p_{12}\ \ p_{22}-1\rangle \\ |p_{12}-2\ \ p_{22}+1\rangle \end{pmatrix} \tag{3.30}$$

where

$$D(-\phi) = \begin{pmatrix} \left(\frac{[p_{12}-p_{22}]}{[2][p_{12}-p_{22}-1]}\right)^{1/2} & -\left(\frac{[p_{12}-p_{22}-2]}{[2][p_{12}-p_{22}-1]}\right)^{1/2} \\ \left(\frac{[p_{12}-p_{22}-2]}{[2][p_{12}-p_{22}-1]}\right)^{1/2} & \left(\frac{[p_{12}-p_{22}]}{[2][p_{12}-p_{22}-1]}\right)^{1/2} \end{pmatrix}. \tag{3.31}$$

Finally, we obtain

$$\begin{aligned} f_2\|p_{12}\ \ p_{22}\rangle &= \left(\frac{[p_{13}-p_{12}+1][p_{12}-p_{23}-1][p_{12}-p_{33}-1][p_{12}-p_{11}]}{[p_{12}-p_{22}][p_{12}-p_{22}+1]}\right)^{1/2} \\ &\quad \times \|p_{12}\ \ p_{22}-1\rangle \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} f_2\|p_{12}-1\ \ p_{22}+1\rangle &= \left(\frac{[p_{13}-p_{22}+1][p_{23}-p_{22}+1][p_{22}-p_{33}-1][p_{11}-p_{22}]}{[p_{12}-p_{22}][p_{12}-p_{22}+1]}\right)^{1/2} \\ &\quad \times \|p_{12}-2\ \ p_{22}+1\rangle \\ &\quad + \left(\frac{[2][p_{13}-p_{12}+1][p_{12}-p_{23}-1][p_{12}-p_{33}-1][p_{12}-p_{11}]}{[p_{12}-p_{22}][p_{12}-p_{22}-2]}\right)^{1/2} \\ &\quad \times |p_{12}-1\ \ p_{22}\rangle\,. \end{aligned} \tag{3.33}$$

We note that the state $|p_{12}-1\ \ p_{22}\rangle$ is of type $(\{1,2\})$. The equations (3.34) and (3.35) correspond, respectively, to the transitions $(\{1\},\{2\}) \to (\{1\},\{2\})$ and $(\{1\},\{2\}) \to (\{1\},\{2\}) + (\{1,2\})$. For example, this method describes successfully the representations of $\mathcal{U}_q(sl(3))$ of dimension 15 for $m = 3$ ($p_{13} = 5,\ p_{23} = 2,\ p_{33} = 0$).

### *3.2. A example of explicit construction of flat representations*

Flat representations were first discovered by Dobrev in [3–5]. These representations have also been studied in [1, 11, 12]. Here we are interested by the explicit construction of flat representations using the primitive Gelfand–Zetlin pattern. Our aim is to show that the Gelfand–Zetlin basis is very adapted for the situation.

In [1], we have introduced some parameters to *break the symmetry between the actions of $e_l$ and $f_l$*. They were taken to be 0, $\frac{1}{2}$ or 1. A good choice of these parameters permits the elimination of the singular vectors (these singular vectors are states arising in the right-hand side of (2.7) but do not obey the triangular inequalities) in a natural way. If an equal number of factors in numerators and denominators are simultaneously equal to zero, and if the vector from the right-hand side of (2.7) is a singular vector, we can adjust these parameters such that the number of zeros in the numerator is greater than the number of zeros in the denominator. This procedure successfully describes the *flat* representations, i.e. when

$$p_{1N} - p_{NN} = m + 1\,. \tag{3.34}$$

For example, the representations of $\mathcal{U}_q(sl(3))$ of dimension 7 for $m = 3$ ($p_{13} = 4$, $p_{23} = 2$, $p_{33} = 0$) is described by

$$\begin{aligned}
f_1|p\rangle &= ([p_{11} - p_{22} - 1])^{1/2}|p_{11} - 1\rangle \\
f_2|p\rangle &= \left(\frac{[p_{13} - p_{12} + 1][p_{12} - p_{23} - 1][p_{12} - p_{33} - 1]}{[p_{12} - p_{22} - 1][p_{12} - p_{22}]}\right)^{1/2}[p_{12} - p_{11}]|p_{12} - 1\rangle \\
&\quad + \left(\frac{[p_{13} - p_{22} + 1][p_{23} - p_{22} + 1][p_{11} - p_{22}]}{[p_{12} - p_{22} + 1][p_{12} - p_{22}]}\right)^{1/2}[p_{22} - p_{33} - 1]|p_{22} - 1\rangle
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
e_1|p\rangle &= [p_{12} - p_{11}]\Big([p_{11} - p_{22}]\Big)^{1/2}|p_{11} + 1\rangle \\
e_2|p\rangle &= \left(\frac{[p_{13} - p_{12}][p_{12} - p_{23}][p_{12} - p_{33}]}{[p_{12} - p_{22} + 1][p_{12} - p_{22}]}\right)^{1/2}|p_{12} + 1\rangle \\
&\quad + \left(\frac{[p_{13} - p_{22}][p_{23} - p_{22}][p_{11} - p_{22} - 1]}{[p_{12} - p_{22} - 1][p_{12} - p_{22}]}\right)^{1/2}|p_{22} + 1\rangle
\end{aligned}$$

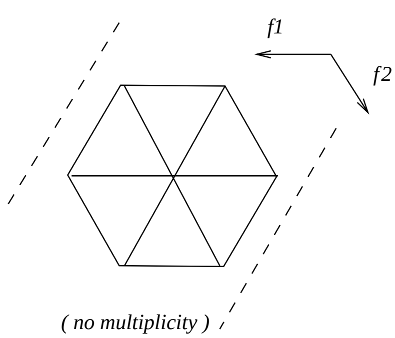


**Figure 3.** Representations of $\mathcal{U}_q(sl(3))$ of dimension 7 for $m = 3$.

we remark that

$$f_1 \left| \begin{matrix} 4 & & 2 & & 0 \\ & 3 & & 2 & \\ & & 3 & & \end{matrix} \right\rangle = f_2 \left| \begin{matrix} 4 & & 2 & & 0 \\ & 3 & & 2 & \\ & & 3 & & \end{matrix} \right\rangle = 0 \tag{3.36}$$

and

$$e_1 \left| \begin{matrix} 4 & & 2 & & 0 \\ & 3 & & 2 & \\ & & 3 & & \end{matrix} \right\rangle = e_2 \left| \begin{matrix} 4 & & 2 & & 0 \\ & 3 & & 2 & \\ & & 3 & & \end{matrix} \right\rangle = 0 \tag{3.37}$$

the others states form just the irreducible representations of dimension 7 (*no multiplicity*). A similar method also describes the representations of dimension 18 for $m = 5$ ($p_{13} = 6$, $p_{23} = 2$, $p_{33} = 0$) (see the figure in [11]).

## Acknowledgments

I would like to thank Daniel Arnaudon and Amitabha Chakrabarti for important and numerous discussions and encouragements during this work. I also thank V K Dobrev for the communication of paper [6].

## References


[1] Abdesselam B, Arnaudon D and Chakrabarti A Representations of $\mathcal{U}_q(sl(N))$ at roots of unity *Preprint* q-alg/9504006, to be published

[2] Arnaudon D and Chakrabarti A 1991 Periodic and partially periodic representations of $SU(N)_q$ *Commun. Math. Phys.* **139** 461

[3] Luszting G 1990 Quantums groups at roots of 1 *Geom. Ded.* **35** 89

[4] Dobrev V K 1991 *Proc. Int. Group Theory Conference (St Andrews 1989)* vol 1 ed Campbell and Robertson (*London Math. Soc. Lect. Notes Series 159*) (Cambridge: Cambridge University Press)

[5] Dobrev V K 1991 Representations of quantum groups *Proc. Symmetries in Science V: Algebraic Structures, Their Representations, Realizations and Physical Applications (Schloss Hofen 1990)* ed B Gruber, L C Biedenharn and H-D Doebner (New York: Plenum) pp 93–135

[6] Dobrev V K 1993 Quantum algebras: representations and real forms *Symp. Symmetries in Science VI: From the Rotation Group to Quantum Algebras (in honour of Professor L C Biedenharn on the occasion of his 70th birthday (Bregenz, 1992))* ed B Gruber (New York: Plenum) Invited talk pp 195–215

[7] Burdík Č, King R C and Welsh T A 1993 The explicit construction of irreducible representations of the quantum algebras $\mathcal{U}_q(sl(N))$ *Proc. 3rd Wigner Symp. (Oxford) Classical and Quantum Systems* ed L L Boyle and A I Solomon (Singapore: World Scientific)

[8] Drinfeld V G 1986 Quantum groups *Proc. Int. Congress of Mathematicians (Berkeley, CA)* vol 1 (New York: Academic) p 798

[9] Jimbo M 1985 $q$-difference analogue of $\mathcal{U}_q(\mathcal{G})$ and the Yang–Baxter equation *Lett. Math. Phys.* **10** 63

[10] Gelf́and I M and Testlin M L 1950 *Dokl. Akad. Nauk SSSR* **71** 825–8

[11] Arnaudon D 1990 Periodic and flat irreducible representations of $SU(3)_q$ *Commun. Math. Phys.* **134** 523

[12] Arnaudon D and Chakrabarti A 1991 Flat periodic representations of $\mathcal{U}_q(\mathcal{G})$ *Commun. Math. Phys.* **139** 605